Report On Light Velocity Gradients

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1 Problem Description

Consider a simple scene with a floating sphere, a camera in front of the sphere, and a point light on the top of the sphere. The camera sees two boundary curves: one curve has the camera rays tangential to the sphere, and the other curve has the shadow rays tangential to the sphere. If we move the point light, then the camera ray curve remains fixed, while the shadow ray curve moves. In this report, we would like to derive the velocity of the shadow ray curve.

2 Methods

2.1 Shadow Ray Curve Velocity

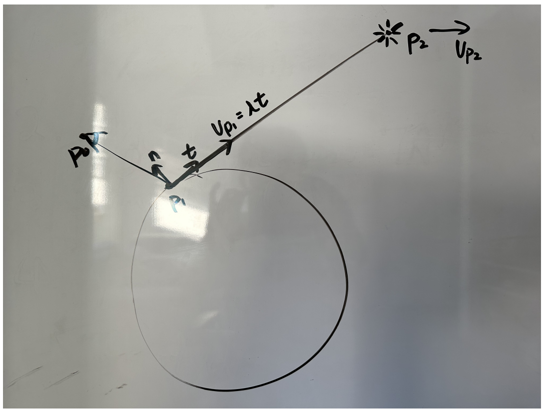


Figure 1: Illustration of intersecting boundary path

Suppose we have a boundary path (x_0, x_1, x_2), where x_0 is the camera, x_1 is a point on the sphere, x_2 is the point light, and $\overrightarrow{x_1x_2}$ is tangential to the sphere. Let $f(x)$ denote the SDF value of point $x \in \mathbb{R}^2$, and $n(x) = \nabla f(x)$ denote the normal. Then a boundary point x_1 should always satisfy

$$\overrightarrow{x_1x_2} \cdot n(x_1) = 0 \tag{1}$$

Now let us take the initial setting as the material space (denoted with p) and take into account the light motion. If the point light moves by velocity v_{x_2}, then we have

$$\overrightarrow{x_1x_2} = \overrightarrow{p_1p_2} + \Delta t v_{p_2} \tag{2}$$

$$n(x_1) = n(p_1) + Hf(p_1)(\Delta t v_{p_1}) + O(|\Delta t|^2) \tag{3}$$

Here Hf is the Hessian matrix and $Hf(p_1)(v)$ describes the change of the normal at p_1 in direction v. Δt is a small time interval. Note that v_{p_1} aligns with $\overrightarrow{p_1p_2}$, so we can write $v_{p_1} = \lambda t$ and $\overrightarrow{p_1p_2} = |p_1 - p_2|t$, where t is a unit tangent vector. Plug everything into equation (1) and we have

$$(|p_1 - p_2|t + \Delta t v_{p_2}) \cdot (n(p1) + Hf(p_1)(\Delta t \lambda t)) = 0 \tag{4}$$

Since $n(p_1) \cdot t = 0$, we can expand and solve

$$\Delta t \lambda = -\frac{\Delta t v_{p_2} \cdot n(p1)}{(|p_1 - p_2|t + \Delta t v_{p_2}) Hf(p_1)(t)} \tag{5}$$

As $\Delta t \to 0$, $\Delta t v_{p_2}$ in the denominator vanishes and we are left with

$$\lambda = -\frac{v_{p_2} \cdot n(p1)}{|p_1 - p_2| \mathbf{K}_t} \tag{6}$$

Here $\mathbf{K}_t = tHf(p_1)(t)$ is the normal curvature at p_1 in direction t.

2.2 Self-Occlusion Curve Velocity

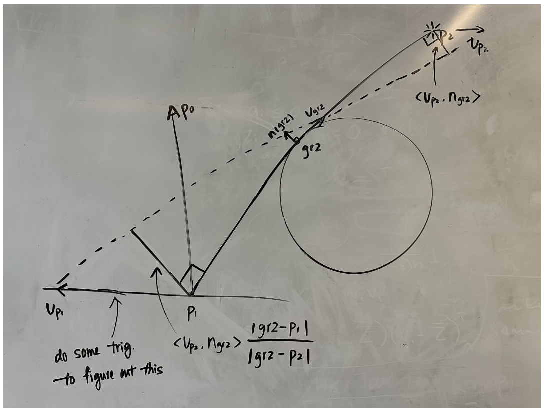


Figure 2: Illustration of grazing boundary path

Recall that we have three types of boundary paths: the incoming intersecting boundary paths, the outcoming intersecting boundary paths, and the grazing boundary paths. For light paths of one bounce, the incoming intersecting boundary paths have their camera rays tangential to the object, the outgoing intersecting boundary paths have their shadow rays tangential to the object, and the grazing boundary paths are due to self-occlusion (visibility changes). We call the point where the path is tangential to the object the grazing point.

We know that the light motion does not affect the incoming intersecting boundary paths and affects the outgoing intersecting boundary paths. However, the light contribution of the outgoing intersecting boundary paths is technically 0, so the gradient due to the light motion is very small if not 0 at all. In fact, the larger gradient comes from the grazing boundary paths. This is because the light motion causes the grazing point to move slightly, similar to the outgoing intersecting boundary paths, but this in turn causes the intersection point to move drastically. Since we identify the path space with the object surface, we want the surface velocity at the intersection point. After computing the grazing point velocity, we can use the tangent plane to approximate the neighborhood of the intersection point. We can do a little trigonometry to transform the grazing velocity to the intersection point velocity.

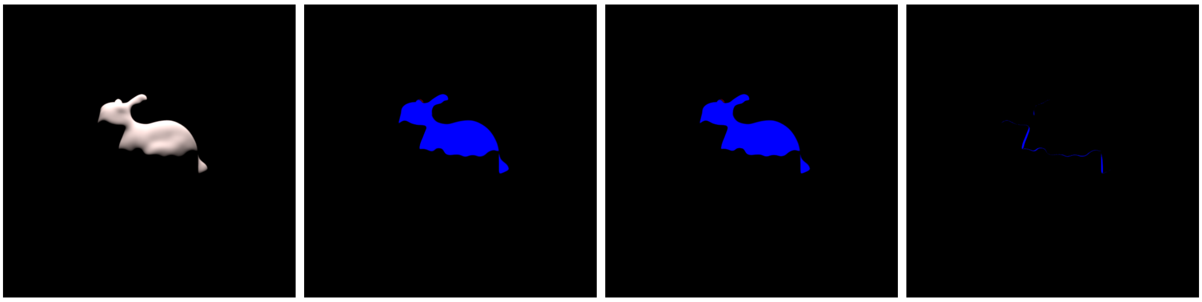
3 Validation & Visualization

3.1 2D Example

Consider a simple 2D scene where we have the camera at $(0, 10)$, a sphere at $(5, 5)$ with radius 2, and a point light at $(10, 7)$. A boundary path would then go from the camera to $(5, 7)$ and then to the point light. If we move the point light down with velocity $(0, -\mu)$, then we can compute everything analytically in equation (4) and solve to get $\lambda = 0.4\mu$. Alternatively, we have $v_{p_2} \cdot n(p1) = (0, -\mu) \cdot (0, 1) = -\mu$, $|p_1 - p_2| = |(5, 7) - (10, 7)| = 5$, and $\mathbf{K}_t$ is the reciprocal of the sphere radius which is 0.5. By equation (6) again we have $\lambda = 0.4\mu$ and $v_{p_1} = (0.4\mu, 0)$.

3.2 3D Example

To visualize the gradients in 3D, we can compute the forward derivatives using finite difference (FD) and automatic differentiation (AD) and then take their difference to show the gradients due to light motion. The gradient resulting from outgoing interacting boundary paths is typically negligible, but if we take a slightly large step in FD, then we can still visualize these gradients and thus show their existence.



(a) Render (b) FD (c) AD (d) FD-AD

Figure 3: Visualization of light motion gradients. Red denotes positive gradients, and blue denotes negative gradients. The two significant regions at the neck and the tail result from grazing boundary paths. Zoom in to see the very small gradients at the boundary of the lighted/dark regions